Differential Geometry Chapter 4

Surfaces in $\mathbb{R}^3$.

Definition 1 *A* ***co-ordinate patch of class*** C^∞ *is a pair* $(U, \mathbf{x})$*, where* U *is an open subset of* $\mathbb{R}^2$ *and* $\mathbf{x} : U \to \mathbb{R}^3$ *a mapping such that*

i. $\mathbf{x}$ *is of class* C^∞,

ii. $\mathbf{x}$ *is 1-1 and* $\mathbf{x}^{-1} : \operatorname{Im} \mathbf{x} \to U$ *is continuous,*

iii. The Jacobian matrix $J\mathbf{x}(\mathbf{u})$ *is of full rank for all* $\mathbf{u} \in U$.

Definition 2 *A* ***surface*** *in* $\mathbb{R}^3$ *(of class* C^∞*) is a non-empty subset* $M \subseteq \mathbb{R}^3$ *such that for each* $\mathbf{p} \in M$ *there exists a co-ordinate patch of class* C^∞, $(U, \mathbf{x})$, *with* $\mathbf{p} \in \mathbf{x}(U) \subseteq M$.

So surfaces can be described by a union of $\{(x_1(\mathbf{u}), x_2(\mathbf{u}), x_3(\mathbf{u})) : \mathbf{u} \in U\}$. Particular examples are $\{(u, v, f(\mathbf{u})) : \mathbf{u} \in U\}$, for $f : U \to \mathbb{R}$, known as Monge patches. This gives the surface $z = f(x, y)$.

It can be shown (see the course on Calculus of Several Variables) that sets of the form $\{\mathbf{x} \in \mathbb{R}^3 : g(\mathbf{x}) = c\}$, when non-empty and where g satisfies $Jg(\mathbf{x}) \neq \mathbf{0}$, defines a surface. We say the surface is defined **implicitly**.

Definition 3 *Let* $\alpha(u) = (g(u), h(u), 0), u \in I$, *be a curve in* $\mathbb{R}^3$ *with* $h(u) > 0$. *Then*

$$\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v)$$

is the ***surface of revolution about the*** x*-axis.*

Exercise 4 *The curve* $\alpha(u) = (r \sin u, R + r \cos u, 0), u \in I$, *with* $R > r > 0$, *gives the torus*

$$\mathbf{x}(u, v) = (r \sin u, (R + r \cos u) \cos v, (R + r \cos u) \sin v).$$

Example 5 *For real* a, b, c *not equal to* 0, *let*

$$M = \left\{ \mathbf{x} \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}.$$

Since $(a,0,0) \in M$ *we have* $M \neq \phi$. *Also*

$$Jg(\mathbf{x}) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$$

which is $= 0$ *only if* $\mathbf{x} = \mathbf{0}$. *But* $\mathbf{0} \notin M$. *Hence* M *is a surface.*

Let $M \subseteq \mathbb{R}^3$ be a surface and $\mathbf{f} : \mathbb{R}^3 \to M$ a map. Choose $\mathbf{y} \in \mathbb{R}^3$ so $\mathbf{f}(\mathbf{y}) \in M$. By the definition of M there exists a co-ordinate patch $(U, \mathbf{x})$, where $U \subseteq \mathbb{R}^2$, $\mathbf{x}(U) \subseteq M$ and $\mathbf{f}(\mathbf{y}) \in \mathbf{x}(U)$. Hence $\mathbf{x}^{-1}\mathbf{f}(\mathbf{y}) \in U$.

To accommodate general situations in which M is not necessarily a subset of a Euclidean Space we make the following definition.

Definition 6 $\mathbf{f} : \mathbb{R}^3 \to M$ *is* ***differentiable in*** M ***at*** $\mathbf{y}$ *provided* $\mathbf{x}^{-1}\mathbf{f} : \mathbb{R}^3 \to U \subseteq \mathbb{R}^2$ *is* C^∞.

But in this course $M \subseteq \mathbb{R}^3$ so we already have a definition of $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ being C^∞. It can be shown that these definitions are equivalent.

In particular, a curve $\alpha : I \to \mathbb{R}^3$ is differentiable in M provided α is a curve, which implies α is C^∞, and lies in M.

Definition 7 *Let* $\mathbf{p} \in M$ *and* $(U, \mathbf{x})$ *a co-ordinate patch such that* $\mathbf{p} \in \mathbf{x}(U) \subseteq M$. *So* $\mathbf{x}^{-1}(\mathbf{p}) = \mathbf{u}_0 \in U$. *Then*

$u \mapsto \mathbf{x}(u, v_0)$, *defined for* $\{u : (u, v_0) \in U\}$, *is the* u-***parameter curve*** *in* M,

$v \mapsto \mathbf{x}(u_0, v)$, *defined for* $\{v : (u_0, v) \in U\}$, *is the* v-***parameter curve*** *in* M.

Example 8 *The surface* $w = \ell_1 u^2 + \ell_2 v^2$ *is defined by the one Monge patch*

$$\left\{ \left(u,\, v,\, \ell_1 u^2 + \ell_2 v^2 \right) : (u, v) \in \mathbb{R}^2 \right\}.$$

The parameter curves are $u \mapsto (u,\, v_0,\, \ell_1 u^2 + \ell_2 v_0^2)$ *and* $v \mapsto (u_0,\, v,\, \ell_1 u_0^2 + \ell_2 v^2)$.

Definition 9 *Let* $\mathbf{p} \in M$. *A vector* $\mathbf{v_p}$ *in* $\mathbb{R}^3$ ***is a tangent vector of*** M *at* $\mathbf{p}$ *if* $\mathbf{v_p}$ *is a velocity vector of some curve in* M.

The set of all tangent vectors of M *at* $\mathbf{p}$ *is the* ***Tangent Space of*** M ***at*** $\mathbf{p}$ *denoted by* $T_{\mathbf{p}}(M)$.

Examples of tangent vectors at $\mathbf{x}(\mathbf{u}) = \mathbf{p}$ would be $\mathbf{x}_u(\mathbf{u})$ and $\mathbf{x}_v(\mathbf{u})$. These are the two columns of $J\mathbf{x}(\mathbf{u})$ and, since this is assumed to be of rank 2, the $\mathbf{x}_u(u)$ and $\mathbf{x}_v(u)$ are linearly independent. Thus

Theorem 10 *Let $\mathbf{p} \in M$ and $(U, \mathbf{x})$ a co-ordinate patch with $\mathbf{x}(\mathbf{u}) = \mathbf{p}$. Then $\mathbf{v}_\mathbf{p} \in T_\mathbf{p}(M)$ if, and only if, $\mathbf{v}_\mathbf{p}$ can be written as a linear combination of $\mathbf{x}_u(\mathbf{u})$ and $\mathbf{x}_v(\mathbf{u})$.*

Proof not given. See Calculus of Several Variables. ■

This result means that $T_\mathbf{p}(M)$ is of dimension 2 and $\mathbf{x}_u(\mathbf{u}) \times \mathbf{x}_v(\mathbf{u})$ is orthogonal to $T_\mathbf{p}(M)$.

Definition 11 *A **vector field** Y **on** M is a function assigning to each $\mathbf{p} \in M$ a tangent vector $Y(\mathbf{p}) \in T_\mathbf{p}(\mathbb{R}^3)$ (not $T_\mathbf{p}(M)$). A vector field Y on M is a **normal vector field** on M if, for each $\mathbf{p} \in M, Y(\mathbf{p})$ is orthogonal to $T_\mathbf{p}(M)$.*

For example, $\mathbf{x}_u \times \mathbf{x}_v$ is a normal vector field. Or

Lemma 12 *If $M = \{\mathbf{x} : g(\mathbf{x}) = c\}$ for some differentiable $g : \mathbb{R}^3 \to \mathbb{R}$, then the gradient ∇g is a normal vector field on M.*

Proof Let $\mathbf{v}_\mathbf{p} \in T_p(M)$. The there exists a curve $\alpha : (-\eta, \eta) \to M$ such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}_\mathbf{p}$. Since $\alpha(t) \in M$ we have $g(\alpha(t)) = c$. The Chain Rule gives

$$\nabla g(\alpha(0)) \bullet \alpha'(0) = 0, \quad \text{i.e.} \quad \nabla g(\mathbf{p}) \bullet \mathbf{v}_\mathbf{p} = 0.$$

Thus $\nabla g(\mathbf{p})$ is orthogonal to $\mathbf{v}_\mathbf{p}$. True for all $\mathbf{v}_\mathbf{p} \in T_\mathbf{p}(M)$ means $\nabla g(\mathbf{p})$ is orthogonal to $T_\mathbf{p}(M)$. True for all $\mathbf{p} \in M$ means ∇g is a normal vector field on M. ■

Example 13 *Returning to $M = \{(u, v, \ell_1 u^2 + \ell_2 v^2) : (u, v) \in \mathbb{R}^2\}$, we have*

$$\mathbf{x}_u(u) = (1, 0, 2\ell_1 v)_{\mathbf{x}(\mathbf{u})} \quad \textit{and} \quad \mathbf{x}_v(u) = (0, 1, 2\ell_2 v)_{\mathbf{x}(\mathbf{u})}$$

in which case

$$\mathbf{x}_u(u) \times \mathbf{x}_v(u) = (1, 0, 2\ell_1 v)_{\mathbf{x}(\mathbf{u})} \times (0, 1, 2\ell_2 v)_{\mathbf{x}(\mathbf{u})} = (-2\ell_1 v, -2\ell_2 v, 1)_{\mathbf{x}(\mathbf{u})}$$

is a normal vector field on M.

Example 14 *Returning to* $M = \{\mathbf{x} \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ *the normal at* $\mathbf{p} \in M$ *is*

$$\left(\frac{2p_1}{a^2}, \frac{2p_2}{b^2}, \frac{2p_3}{c^2}\right)_{\mathbf{p}}.$$

Question How does the unit normal vector change as $\mathbf{p}$ changes?

Definition 15 *If* $\mathbf{p} \in M$*, for each* $\mathbf{v_p} \in T_{\mathbf{p}}(M)$ *defined*

$$S_{\mathbf{p}}(\mathbf{v_p}) = -\nabla_{\mathbf{v_p}} U$$

where U *is a unit normal vector field.* $S_{\mathbf{p}}$ *is called the* ***Shape Operator*** *of* M *at* $\mathbf{p}$*.*

To recall, $\nabla_{\mathbf{v_p}} U = U'(\mathbf{p} + t\mathbf{v})|_{t=0}$, i.e. the initial rate of change of U as you move from $\mathbf{p}$ in the direction of $\mathbf{v}$. Also

$$\nabla_{\mathbf{v_p}} V = \sum_{i=1}^{n} \mathbf{v_p}[f_i]\, U_i(\mathbf{p}) \tag{1}$$

where, similarly, $\mathbf{v_p}[f] = f'(\mathbf{p} + t\mathbf{v})_{t=0}$ is the initial rate of change of the scalar-valued function f as you move from $\mathbf{p}$ is the direction of $\mathbf{v}$. It was shown that $\mathbf{v_p}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{v_p}$, i.e. the component of the gradient in the direction of travel.

Lemma 16 *For each* $\mathbf{p} \in M$*, the operator* $S_{\mathbf{p}} : T_{\mathbf{p}}(M) \to T_{\mathbf{p}}(M)$ *is linear.*

Proof Let $\mathbf{v_p} \in T_{\mathbf{p}}(M)$. Since $U \bullet U = 1$ we have $\mathbf{v_p}[U \bullet U] = 0$ (there is no change as we move). From an earlier lemma, $\mathbf{v_p}[U \bullet U] = 2\nabla_{\mathbf{v_p}} U \bullet U(\mathbf{p}) = -S_{\mathbf{p}}(\mathbf{v_p}) \bullet U(\mathbf{p})$. Thus $S_{\mathbf{p}}(\mathbf{v_p}) \bullet U(\mathbf{p}) = 0$ which means that $S_{\mathbf{p}}(\mathbf{v_p}) \in T_{\mathbf{p}}(M)$. Hence $S_{\mathbf{p}} : T_{\mathbf{p}}(M) \to T_{\mathbf{p}}(M)$.

To see it is linear let $\mathbf{v_p}, \mathbf{w_p} \in T_{\mathbf{p}}(M)$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned} S_{\mathbf{p}}(\lambda\mathbf{v_p} + \mu\mathbf{w_p}) &= -\nabla_{\lambda\mathbf{v_p} + \mu\mathbf{w_p}} U \\ &= -\left(\lambda\nabla_{\mathbf{v_p}} U + \mu\nabla_{\mathbf{w_p}} U\right) \\ &\quad \text{by earlier lemma} \\ &= \lambda S_{\mathbf{p}}(\mathbf{v_p}) + \mu S_{\mathbf{p}}(\mathbf{w_p}). \end{aligned}$$

Thus the linearity of $S_{\mathbf{p}}$ follows from the linearity of the covariant derivative.

■

If $S_{\mathbf{p}}$ is measuring the instantaneous change of the normal it is therefore measuring any change in the tangent space and thus of the surface.

We will denoted by S both the set of all S_p for all p or for a representative element from this set.

Question How to calculate $S_{\mathbf{p}}$? It is a linear transformation and so it is sufficient to calculate it's effect on a basis. In the terms of $\mathbf{p}$ in a patch $(U, \mathbf{x})$ and basis would be $\{\mathbf{x}_u, \mathbf{x}_v\}$. There is a possible confusion here between an open set $U \subseteq \mathbb{R}^2$ and a normal vector field U. So now let N denote a unit normal vector field on a surface.

Lemma 17 *Let* $\mathbf{p} \in M$ *and* $(U, \mathbf{x})$ *a co-ordinate patch such that* $\mathbf{p} \in \mathbf{x}(U) \subseteq M$. *So* $\mathbf{p} = \mathbf{x}(\mathbf{u}_0)$ *for some* $\mathbf{u}_0 \in U$. *Let* Z *be a vector field on* $\mathbf{x}(U)$, *so* $Z = \sum_{i=1}^{3} y_i U_i$, *where* $y : \mathbf{x}(U) \to \mathbb{R}$. *Define* $z_i(u) = y_i(\mathbf{x}(u)), 1 \le i \le 3$, *so* $Z = \sum_{i=1}^{3} z_i U_i$. *Then*

$$\begin{aligned} \nabla_{\mathbf{x}_u}(Z) &= \sum_{i=1}^{3} \frac{\partial z_i}{\partial u}(\mathbf{u}_0)\, U_i(\mathbf{p}), \\ \nabla_{\mathbf{x}_v}(Z) &= \sum_{i=1}^{3} \frac{\partial z_i}{\partial v}(\mathbf{u}_0)\, U_i(\mathbf{p}), \end{aligned}$$

Proof By (1) above

$$\nabla_{\mathbf{x}_u}(Z) = \sum_{i=1}^{3} \mathbf{x}_u[y_i]\, U_i(\mathbf{p})$$

Here

$$\mathbf{x}_u[y_i] = \nabla y_i(\mathbf{p}) \bullet \mathbf{x}_u = \sum_{j=1}^{3} \frac{\partial yi}{\partial x_j}(\mathbf{p}) \frac{\partial x_j}{\partial u}(\mathbf{u}_0) = \frac{\partial z}{\partial u}(\mathbf{u}_0)$$

by the Chain Rule. This gives the first stated result. There are no new ideas in the proof of the second statement.

■

Lemma 18 *Notation as before. With N a unit vector field on $\mathbf{x}(U)$.*

$$\begin{aligned}
S_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_{\mathbf{u}} &= N \bullet \mathbf{x}_{uu} \\
S_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_v &= N \bullet \mathbf{x}_{uv} \\
S_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_u &= N \bullet \mathbf{x}_{vu} \\
S_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_v &= N \bullet \mathbf{x}_{vv}.
\end{aligned}$$

(What is meant here is that given $\mathbf{p} \in \mathbf{x}(U)$ there exists $\mathbf{u}_0 : x(\mathbf{u}_0) = \mathbf{p}$. Then $S_{\mathbf{p}}(\mathbf{x}_u(\mathbf{u}_0)) \bullet \mathbf{x}_{\mathbf{u}}(\mathbf{u}_0) = N(\mathbf{p}) \bullet \mathbf{x}_{uu}(\mathbf{u}_0)$, etc.)

Proof Choose $Z = N$ in the previous lemma so $N = \sum_{i=1}^{3} z_i U_i$ and

$$\nabla_{\mathbf{x}_{\mathbf{u}}}(N) = \sum_{i=1}^{3} \frac{\partial z_i}{\partial u}(\mathbf{u}_0)\, U_i(\mathbf{p})\,.$$

But for each $\mathbf{p} \in \mathbf{x}(U)$, the tangent vector $\mathbf{x}_u(\mathbf{p})$ lies in $T_{\mathbf{p}}(M)$ and so is orthogonal to N, i.e.

$$0 = \mathbf{x}_u \bullet N = \sum_{i=1}^{3} \frac{\partial x_i}{\partial u}\, z_i.$$

Differentiating w.r.t. u,

$$0 = \sum_{i=1}^{3} \frac{\partial^2 x_i}{\partial u^2}\, z_i + \sum_{i=1}^{3} \frac{\partial x_i}{\partial u}\frac{\partial z_i}{\partial u} = N \bullet \mathbf{x}_{uu} + \mathbf{x}_u \bullet \nabla_{\mathbf{x}_{\mathbf{u}}}(N)\,,$$

which gives the first result. The other three follow similarly. ■

Remark Since $\mathbf{x}$ is a C^∞ map it can be shown (see Calculus of Several Variables) that $\mathbf{x}_{uv} = \mathbf{x}_{vu}$. Thus

$$S_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_v = N \bullet \mathbf{x}_{uv} = N \bullet \mathbf{x}_{vu} = S_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_u. \tag{2}$$

In fact this result holds for all pairs of vectors in the Tangent Space.

Lemma 19 *For all $\mathbf{u}_{\mathbf{p}}, \mathbf{v}_{\mathbf{p}} \in T_{\mathbf{p}}(M)$ we have $S_{\mathbf{p}}(\mathbf{u}_{\mathbf{p}}) \bullet \mathbf{v}_{\mathbf{p}} = S_{\mathbf{p}}(\mathbf{v}_{\mathbf{p}}) \bullet \mathbf{u}_{\mathbf{p}}$.*

Proof $\mathbf{u_p} = k\mathbf{x}_u + l\mathbf{x}_v$ and $\mathbf{v_p} = s\mathbf{x}_u + t\mathbf{x}_v$. Then

$$\begin{aligned}
S_{\mathbf{p}}(\mathbf{u_p}) \bullet \mathbf{v_p} &= S_{\mathbf{p}}(k\mathbf{x}_u + l\mathbf{x}_v) \bullet (s\mathbf{x}_u + t\mathbf{x}_v) \\
&= (kS_{\mathbf{p}}(\mathbf{x}_u) + lS_{\mathbf{p}}(\mathbf{x}_v)) \bullet (s\mathbf{x}_u + t\mathbf{x}_v) \\
&= ksS_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_u + ktS_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_v + lsS_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_u + ltS_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_v \\
&= ksS_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_u + ktS_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_u + lsS_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_v + ltS_{\mathbf{p}}(\mathbf{x}_v) \bullet \mathbf{x}_v \\
&\quad \text{having used (2)} \\
&= S_{\mathbf{p}}(s\mathbf{x}_u) \bullet k\mathbf{x}_u + S_{\mathbf{p}}(t\mathbf{x}_v) \bullet k\mathbf{x}_u + S_{\mathbf{p}}(s\mathbf{x}_u) \bullet l\mathbf{x}_v + S_{\mathbf{p}}(t\mathbf{x}_v) \bullet l\mathbf{x}_v \\
&= (S_{\mathbf{p}}(s\mathbf{x}_u) + S_{\mathbf{p}}(t\mathbf{x}_v)) \bullet k\mathbf{x}_u + (S_{\mathbf{p}}(s\mathbf{x}_u) + S_{\mathbf{p}}(t\mathbf{x}_v)) \bullet l\mathbf{x}_v \\
&= (S_{\mathbf{p}}(s\mathbf{x}_u) + S_{\mathbf{p}}(t\mathbf{x}_v)) \bullet (k\mathbf{x}_u + l\mathbf{x}_v) \\
&= S_{\mathbf{p}}(s\mathbf{x}_u + t\mathbf{x}_v) \bullet (k\mathbf{x}_u + l\mathbf{x}_v) \\
&= S_{\mathbf{p}}(\mathbf{v_p}) \bullet \mathbf{u_p}.
\end{aligned}$$

■

This results says that $S_{\mathbf{p}}$ is a **symmetric operator**.

Given a patch $(U, \mathbf{x})$ it is easy to calculate $N \bullet \mathbf{x}_{uu}$,... etc but how can we use these to say something about the surface? One problem is that though $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis the vectors may not be orthogonal.

If $\left\{\mathbf{e}^1_{\mathbf{p}}, \mathbf{e}^2_{\mathbf{p}}\right\}$ is an orthonormal basis of $T_{\mathbf{p}}(M)$ then the matrix of $S_{\mathbf{p}}$ w.r.t. this basis is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

where $S_{\mathbf{p}}\left(\mathbf{e}^1_{\mathbf{p}}\right) = a\mathbf{e}^1_{\mathbf{p}} + b\mathbf{e}^2_{\mathbf{p}}$ and $S_{\mathbf{p}}\left(\mathbf{e}^2_{\mathbf{p}}\right) = c\mathbf{e}^1_{\mathbf{p}} + d\mathbf{e}^2_{\mathbf{p}}$. But since $S_{\mathbf{p}}$ is symmetric

$$c = \mathbf{e}^1_{\mathbf{p}} \bullet S_{\mathbf{p}}\left(\mathbf{e}^2_{\mathbf{p}}\right) = \mathbf{e}^2_{\mathbf{p}} \bullet S_{\mathbf{p}}\left(\mathbf{e}^1_{\mathbf{p}}\right) = b.$$

Thus the matrix is also symmetric,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

The eigenvalues of this are solutions of

$$x^2 - (a+d)\,x - \left(b^2 - ad\right) = 0.$$

The solutions are real since the discriminant citifies

$$(a+d)^2 + 4\left(b^2 - ad\right) = (a-d)^2 + 4b^2 \geq 0.$$

(This is the virtue of the matrix being symmetric).

Let $k_1(\mathbf{p}), k_2(\mathbf{p})$ be these eigenvalues.

If $k_1 = k_2$ then $S_{\mathbf{p}}(\mathbf{v}_{\mathbf{p}}) = k\mathbf{v}_{\mathbf{p}}$ for all $\mathbf{v}_{\mathbf{p}} \in T_{\mathbf{p}}(M)$ where k is this common value. In this case we say that $\mathbf{p}$ is an **umbilic point.**

If $k_1 \neq k_2$ there exists, for each k_i, an eigenvector $\mathbf{v}_i = \mathbf{v}_i(\mathbf{p})$ say, which we can assume are unit length.

Note that

$$\begin{aligned} k_1\mathbf{v}_1 \bullet \mathbf{v}_2 &= S_{\mathbf{p}}(\mathbf{v}_1) \bullet \mathbf{v}_2 = \mathbf{v}_1 \bullet S_{\mathbf{p}}(\mathbf{v}_2) \quad \text{since } S_{\mathbf{p}} \text{ is symmetric} \\ &= k_2\mathbf{v}_1 \bullet \mathbf{v}_2. \end{aligned}$$

Since $k_1 \neq k_2$ we must have $\mathbf{v}_1 \bullet \mathbf{v}_2 = 0$, i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

Further,

$$k_1 = k_1\mathbf{v}_1 \bullet \mathbf{v}_1 = S_{\mathbf{p}}(\mathbf{v}_1) \bullet \mathbf{v}_1 = -\nabla_{\mathbf{v}_1} N \bullet \mathbf{v}_1.$$

If $k_1 > 0$ then, as we move from $\mathbf{p}$ in the $\mathbf{v}_1$ direction $\nabla_{\mathbf{v}_1} N < 0$, i.e. the change in the normal has a component in the $-$ve $\mathbf{v}_1$ direction. That means the curve 'curves up to' the normal. If $k_1 < 0$ then the curve 'down and away' from the normal.

Definition 20 *The* $k_1(\mathbf{p}), k_2(\mathbf{p})$ *are the* ***principal curvatures*** *and* $\mathbf{v}_1(\mathbf{p})$ *and* $\mathbf{v}_2(\mathbf{p})$ *the* ***principal vectors*** *of* M *at* $\mathbf{p}$.

Note that, w.r.t $\mathbf{v}_1$ and $\mathbf{v}_2$, the Shape Operator $S_{\mathbf{p}}$ has the associated matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

which has determinant k_1k_2 and trace $k_1 + k_2$.

Linear Algebra The determinant and trace is the same for any matrix associated with a given linear map and we can talk of the determinant and trace of a linear map.

Since $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis of $T_{\mathbf{p}}(M)$ there exist e, f, g and $h \in \mathbb{R}$ such that $S_{\mathbf{p}}(\mathbf{x}_u) = e\mathbf{x}_u + f\mathbf{x}_v$ and $S_{\mathbf{p}}(\mathbf{x}_v) = g\mathbf{x}_u + h\mathbf{x}_v$. Then, w.r.t this basis, $S_{\mathbf{p}}$ is represented by the matrix

$$\begin{pmatrix} e & g \\ f & h \end{pmatrix}.$$

By our linear algebra observation we have $eh - fg = k_1k_2$ and $e + h = k_1 + k_2$. But, how to calculate $eh - fg$ and $e + h$?

From Lemma 18 we have

$$N \bullet \mathbf{x}_{uu} = S_{\mathbf{p}}(\mathbf{x}_u) \bullet \mathbf{x}_u = (e\mathbf{x}_u + f\mathbf{x}_v) \bullet \mathbf{x}_{\mathbf{u}} = e\mathbf{x}_u \bullet \mathbf{x}_u + f\mathbf{x}_v \bullet \mathbf{x}_u,$$

with three more to follow. This motivates giving labels as follows:

$$\ell = N \bullet \mathbf{x}_{uu},\ m = N \bullet \mathbf{x}_{uv} = N \bullet \mathbf{x}_{vu} \quad \text{and} \quad n = N \bullet \mathbf{x}_{vv}.$$

Further, set $E = \mathbf{x}_u \bullet \mathbf{x}_u$, $F = \mathbf{x}_v \bullet \mathbf{x}_u = \mathbf{x}_u \bullet \mathbf{x}_v$ and $G = \mathbf{x}_v \bullet \mathbf{x}_v$. So six values to calculate. The results of Lemma 18 become

$$m = eF + fG,\ \ell = eE + fF,\ n = gF + hG \quad \text{and} \quad m = gE + hF.$$

In matrix form,

$$\begin{pmatrix} m & \ell \\ n & m \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} F & E \\ G & F \end{pmatrix}.$$

Taking determinants,

$$k_1k_2 = eh - fg = \frac{m^2 - \ell n}{F^2 - GE}.$$

Solving the matrix equation

$$\begin{aligned} \begin{pmatrix} e & f \\ g & h \end{pmatrix} &= \frac{1}{F^2 - EG} \begin{pmatrix} F & -E \\ -G & F \end{pmatrix} \begin{pmatrix} m & \ell \\ n & m \end{pmatrix} \\ &= \frac{1}{F^2 - EG} \begin{pmatrix} Fm - En & F\ell - Em \\ -Gm + Fn & -G\ell + Fm \end{pmatrix}. \end{aligned}$$

Then

$$k_1 + k_2 = e + h = \frac{2mF - \ell G - En}{F^2 - EG}.$$

Definition 21 *The* ***Gaussian curvature*** *of* M *at* $\mathbf{p}$ *is* $K(\mathbf{p}) = k_1(\mathbf{p}) + k_2(\mathbf{p})$, *the* ***mean curvature*** *of* M *at* $\mathbf{p}$ *is* $H(\mathbf{p}) = \frac{1}{2}(k_1(\mathbf{p}) + k_2(\mathbf{p}))$.

Note we can calculate $K(\mathbf{p})$ and $H(\mathbf{p})$ by the above formula and recover k_1 and k_2 by solving $x^2 - 2Hx + K = 0$.

Recall we assume that k_i and k_2 are non both zero. So if $K = 0$ then either $k_1 = 0$ and $k_2 \neq 0$ or vice versa. In this case the surface "looks like" a cylinder around $\mathbf{p}$ and we say it is **parabolic**.

If $K > 0$ then either $k_1 > 0$, $k_2 > 0$ or $k_1 < 0$, $k_2 < 0$. In both cases the surface bends towards (or away from) the normal, in whichever direction you travel. That is the surface stays to one side of the tangent space. We say M is **elliptic** at p.

If $K < 0$ then either $k_1 > 0$, $k_2 < 0$ or $k_1 < 0$, $k_2 > 0$. So in different directions you bend towards and away from the normal. We say that M is **hyperbolic**, or a **saddle**, near $\mathbf{p}$.

Before we look at calculating these curvatures we state, without proof, a celebrated result dues to Gauss.

Theorem 22 ***Theorema egreguim*** *An isometry preserves the Gaussian Curvature; Let* $\mathbf{F} : S_1 \to S_2$ *be an isometry between two surfaces. For every* $\mathbf{p} \in S_1$ *we have* $K(\mathbf{p}) = K(\mathbf{F}(\mathbf{p}))$.

Example 23 *Find the Gaussian curvature of a surface of revolution*

$$\mathbf{x} : \mathbf{u} \mapsto (g(u), h(u)\cos v, h(u)\sin v).$$

Solution

$$\mathbf{x}_u = (g', h'\cos v, h'\sin v)_{\mathbf{x}(\mathbf{u})} \quad \text{and} \quad \mathbf{x}_v = (0, -h\sin v, h\cos v)_{\mathbf{x}(\mathbf{u})}.$$

Then

$$\begin{aligned} E &= \mathbf{x}_u \bullet \mathbf{x}_u = (g')^2 + (h')^2 , \\ F &= \mathbf{x}_v \bullet \mathbf{x}_u = -h' \cos v h \sin v + h' \sin v h \cos v = 0, \\ G &= \mathbf{x}_v \bullet \mathbf{x}_v = h^2. \end{aligned}$$

So $F^2 - EG = -h^2 \left((g')^2 + (h')^2\right)$.

For the normal vector field

$$\mathbf{x}_u \times \mathbf{x}_v = (h'h,\ -g'h \cos v,\ -g'h \sin v)_{\mathbf{x}(\mathbf{u})} .$$

we have $\|\mathbf{x}_u \times \mathbf{x}_v\| = h \left((h')^2 + (g')^2\right)^{1/2}$. Thus

$$N = \frac{1}{D} (h',\ -g' \cos v,\ -g' \sin v)_{\mathbf{x}(\mathbf{u})} ,$$

where $D = \left((h')^2 + (g')^2\right)^{1/2}$.

Next

$$\begin{aligned} \mathbf{x}_{uu} &= (g'',\ h'' \cos v,\ h'' \sin v)_{\mathbf{x}(\mathbf{u})} , \\ \mathbf{x}_{uv} &= (0, -h' \sin v,\ h' \cos v)_{\mathbf{x}(\mathbf{u})} , \\ \mathbf{x}_{vv} &= (0,\ -h \cos v,\ -h \sin v)_{\mathbf{x}(\mathbf{u})} . \end{aligned}$$

Then

$$\begin{aligned} \ell &= N \bullet \mathbf{x}_{uu} = \frac{1}{D} (h'g'' - g'h'') . \\ m &= N \bullet \mathbf{x}_{uv} = 0, \\ n &= N \bullet \mathbf{x}_{vv} = \frac{1}{D} hg'. \end{aligned}$$

Therefore

$$\begin{aligned} K &= \frac{m^2 - \ell n}{F^2 - GE} = \frac{1}{D} (h'g'' - g'h'') \frac{1}{D} hg' \frac{1}{h^2 \left((g')^2 + (h')^2\right)} \\ &= \frac{g' (h'g'' - g'h'')}{h \left((g')^2 + (h')^2\right)^2}. \end{aligned}$$

■

In particular, for a torus, when $g(u) = r\sin u$ and $h(u) = R + r\cos u$ we find

$$K = \frac{\cos u}{r(R + r\cos u)}.$$

So there are hyperbolic ($\pi/2 < u < 3\pi/2$), parabolic ($u = 0$ or $u = 3\pi/2$) or elliptic ($0 < u < \pi/2, 3\pi/2 < u < 2\pi$) points on a torus.

Example 24 *Find the Gaussian curvature for the graph of* $w = \ell_1 u^2 + \ell_2 v^2$.

Solution we have

$$\begin{aligned} \mathbf{x}_u &= (1,\, 0,\, 2\ell_1 u)_{\mathbf{x(u)}}\,;\ \mathbf{x}_v = (0,\, 1,\, 2\ell_2 u)_{\mathbf{x(u)}}\,;\ \mathbf{x}_{uu} = (0,\, 0,\, 2\ell_1)_{\mathbf{x(u)}}\,, \\ \mathbf{x}_{uv} &= 0 \quad \text{and} \quad \mathbf{x}_{vv} = (0,\, 0,\, 2\ell_2)_{\mathbf{x(u)}}\,. \end{aligned}$$

So

$$N = \frac{1}{D}\left(-2\ell_1 u,\, -2\ell_2 v,\, 1\right)_{\mathbf{x(u)}}\,,$$

where $D = \left(1 + 4\ell_1^2 u^2 + 4\ell_2^2 v^2\right)^{1/2}$. Then $E = 1 + 4\ell_1^2 u^2$; $F = 4\ell_1\ell_2 uv$; $G = 1 + 4\ell_2^2 v^2$; $m = 0$; $\ell = 2\ell_1/D$ and $n = 2\ell_2/D$. Hence

$$K = \frac{4\ell_1\ell_2}{D^4}.$$

■

Lemma 25 *If* $\mathbf{v_p}, \mathbf{w_p} \in T_{\mathbf{p}}(M)$ *are linearly independent tangent vectors then*

$$\begin{aligned} S_{\mathbf{p}}(\mathbf{v_p}) \times S_{\mathbf{p}}(\mathbf{w_p}) &= K(\mathbf{p})\, \mathbf{v_p} \times \mathbf{w_p}, \\ S_{\mathbf{p}}(\mathbf{v_p}) \times \mathbf{w_p} + \mathbf{v_p} \times S_{\mathbf{p}}(\mathbf{w_p}) &= 2H(\mathbf{p})\, \mathbf{v_p} \times \mathbf{w_p}. \end{aligned}$$

Proof Since $\mathbf{v_p}$ and $\mathbf{w_p}$ form a basis for $T_{\mathbf{p}}(M)$ there exist real a, b, c and d such that

$$S_{\mathbf{p}}(\mathbf{v_p}) = a\mathbf{v_p} + b\mathbf{w_p} \quad \text{and} \quad S_{\mathbf{p}}(\mathbf{w_p}) = c\mathbf{v_p} + d\mathbf{w_p}.$$

Then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is the matrix associated with $S_{\mathbf{p}}$ w.r.t this basis. Then

$$K(\mathbf{p}) = ad - bc \quad \text{and} \quad H(\mathbf{p}) = \frac{1}{2}(a+d)\,. \tag{3}$$

But

$$\begin{aligned} S_{\mathbf{p}}(\mathbf{v_p}) \times S_{\mathbf{p}}(\mathbf{w_p}) &= (a\mathbf{v_p} + b\mathbf{w_p}) \times (c\mathbf{v_p} + d\mathbf{w_p}) \\ &= (ad - bc)\,\mathbf{v_p} \times \mathbf{w_p} \\ &= K(\mathbf{p}) \quad \text{by } (3)\,. \end{aligned}$$

Also

$$\begin{aligned} S_{\mathbf{p}}(\mathbf{v_p}) \times \mathbf{w_p} + \mathbf{v_p} \times S_{\mathbf{p}}(\mathbf{w_p}) &= (a\mathbf{v_p} + b\mathbf{w_p}) \times \mathbf{w_p} + \mathbf{v_p} \times (c\mathbf{v_p} + d\mathbf{w_p}) \\ &= a\mathbf{v_p} \times \mathbf{w_p} + d\mathbf{v_p} \times \mathbf{w_p} \\ &= 2H(\mathbf{p})\,\mathbf{v_p} \times \mathbf{w_p}. \end{aligned}$$

■

Definition 26 *Assume V is a tangent vector field on M such that $V(\mathbf{p}) \in T_{\mathbf{p}}(M)$ for all $\mathbf{p} \in M$. Define $S(V)$ a vector field on M by $S(V)(\mathbf{p}) = S_{\mathbf{p}}(V(\mathbf{p}))$.*

The pointwise results of the last lemma can be written in terms of vector fields V and W,

$$\begin{aligned} S(V) \times S(W) &= KV \times W \\ S(V) \times W + V \times S(W) &= 2HV \times W, \end{aligned}$$

where $K, H : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{p} \mapsto K(\mathbf{p})$ and $\mathbf{p} \mapsto H(\mathbf{p})$ respectively.

A result for vectors is the *Lagrange identity*

$$(\mathbf{v} \times \mathbf{w}) \bullet (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{v} \bullet \mathbf{a} & \mathbf{v} \bullet \mathbf{b} \\ \mathbf{w} \bullet \mathbf{a} & \mathbf{w} \bullet \mathbf{b} \end{vmatrix}.$$

This allows us to solve our system as

$$K\|V \times W\|^2 = (V \times W) \bullet (S(V) \times S(W)) = \begin{vmatrix} V \bullet S(V) & V \bullet S(W) \\ W \bullet S(V) & W \bullet S(W) \end{vmatrix}.$$

So

$$K = \frac{1}{\|V \times W\|^2} \begin{vmatrix} V \bullet S(V) & V \bullet S(W) \\ W \bullet S(V) & W \bullet S(W) \end{vmatrix} = \frac{\begin{vmatrix} V \bullet S(V) & V \bullet S(W) \\ W \bullet S(V) & W \bullet S(W) \end{vmatrix}}{\begin{vmatrix} V \bullet V & V \bullet W \\ W \bullet V & W \bullet W \end{vmatrix}}.$$

Similarly

$$H = \frac{1}{2}\left(\frac{\begin{vmatrix} V \bullet S(V) & V \bullet S(W) \\ W \bullet V & W \bullet W \end{vmatrix} + \begin{vmatrix} V \bullet V & V \bullet W \\ W \bullet S(V) & W \bullet S(W) \end{vmatrix}}{\begin{vmatrix} V \bullet V & V \bullet W \\ W \bullet V & W \bullet W \end{vmatrix}} \right).$$

Let Z be a normal vector field on M, normalized as $Z/\|Z\|$. Let V be a tangent vector field on M. Then

$$S(V) = -\nabla_V\left(\frac{Z}{\|Z\|}\right) = -\frac{1}{\|Z\|}\nabla_V(Z) - V\left(\frac{1}{\|Z\|}\right)Z,$$

by a result seen at the end of Chapter 2. Note that the second term is normal to M. Let W be another tangent vector field so

$$S(W) = -\frac{1}{\|W\|}\nabla_W(Z) - W\left(\frac{1}{\|W\|}\right)Z.$$

Now choose $Z = V \times W$ and consider

$$\begin{aligned} S(V) \times S(W) &= \frac{1}{\|Z\|^2}\nabla_V(Z) \times \nabla_W(Z) + \frac{1}{\|Z\|}W\left(\frac{1}{\|W\|}\right)\nabla_V(Z) \times Z \\ &\quad + \frac{1}{\|W\|}V\left(\frac{1}{\|Z\|}\right)Z \times \nabla_W(Z). \end{aligned}$$

Now dot with $V \times W = Z$. Note that $\nabla_V(Z), \nabla_W(Z) \in T_{\mathbf{p}}(M)$ and Z is orthogonal to $T_p(M)$ so $\nabla_V(Z) \times Z$, $Z \times \nabla_W(Z) \in T_{\mathbf{p}}(M)$ (remembering we are in 3 dimensions). Then $(\nabla_V(Z) \times Z) \bullet Z = 0$ and $(Z \times \nabla_W(Z)) \bullet Z = 0$.

From above

$$K\|V \times W\|^2 = (V \times W) \bullet (S(V) \times S(W)) = \frac{1}{\|Z\|^2}(\nabla_V(Z) \times \nabla_W(Z)) \bullet Z$$

So

$$K = \frac{1}{\|Z\|^4}(\nabla_V(Z) \times \nabla_W(Z)) \bullet Z.$$

Example 27 *Find K for*

$$M = \left\{ \mathbf{x} \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \right\}.$$

Solution. As seen earlier, for an implicitly given surface the gradient vector is normal to the Tangent Space so we can choose

$$Z = \frac{1}{2} \nabla g = \sum_{i=1}^{3} \frac{x_i}{a_i^2} U_i.$$

Then

$$\|Z\|^4 = \left(\sum_{i=1}^{3} \frac{x_i^2}{a_i^4} \right)^2.$$

Let $\mathbf{p} \in M$, $V = \sum_{i=1}^{3} v_i U_i$ and $W = \sum_{i=1}^{3} w_i U_i \in T_{\mathbf{p}}(M)$ where $v_i, w_i : \mathbb{R}^3 \to \mathbb{R}$. Then

$$\nabla_V Z = \sum_{i=1}^{3} V \left[\frac{x_i}{a_i^2} \right] U_i = \sum_{i=1}^{3} \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j} \left(\frac{x_i}{a_i^2} \right) U_i = \sum_{i=1}^{3} \frac{v_i}{a_i^2} U_i.$$

Similarly

$$\nabla_W Z = \sum_{i=1}^{3} \frac{w_i}{a_i^2} U_i.$$

Then

$$\begin{aligned} \nabla_V Z \times \nabla_W Z \quad = \quad & \left(\frac{v_2}{a_2^2} \frac{w_3}{a_3^2} - \frac{v_3}{a_3^2} \frac{w_2}{a_2^2} \right) U_1 + \left(\frac{v_3}{a_3^2} \frac{w_1}{a_1^2} - \frac{v_1}{a_1^2} \frac{w_3}{a_3^2} \right) U_2 \\ & + \left(\frac{v_1}{a_1^2} \frac{w_2}{a_2^2} - \frac{v_2}{a_2^2} \frac{w_1}{a_1^2} \right) U_3. \end{aligned}$$

Thus

$$\begin{aligned} Z \bullet \nabla_V Z \times \nabla_W Z &= \frac{x_1}{a_1^2}\left(\frac{v_2}{a_2^2}\frac{w_3}{a_3^2} - \frac{v_3}{a_3^2}\frac{w_2}{a_2^2}\right) + \frac{x_2}{a_2^2}\left(\frac{v_3}{a_3^2}\frac{w_1}{a_1^2} - \frac{v_1}{a_1^2}\frac{w_3}{a_3^2}\right) \\ &\quad + \frac{x_3}{a_3^2}\left(\frac{v_1}{a_1^2}\frac{w_2}{a_2^2} - \frac{v_2}{a_2^2}\frac{w_1}{a_1^2}\right) \\ &= \frac{1}{a_1^2 a_2^2 a_3^2}(x_1, x_2, x_3) \bullet (V \times W) \\ &= \frac{1}{a_1^2 a_2^2 a_3^2}(x_1, x_2, x_3) \bullet Z \\ &= \frac{1}{a_1^2 a_2^2 a_3^2}\sum_{i=1}^{3}\frac{x_i^2}{a_i^2} = \frac{1}{a_1^2 a_2^2 a_3^2}. \end{aligned}$$

Hence

$$K = \frac{1}{a_1^2 a_2^2 a_3^2}\frac{1}{\|Z\|^4} = \frac{1}{a_1^2 a_2^2 a_3^2}\left(\sum_{i=1}^{3}\frac{x_i^2}{a_i^4}\right)^{-2}.$$